International Journal of Theoretical Physics, Vol. 21, No. 5, 1982

Equation of Geodesic Deviation and Killing Tensors

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Received February 12, 1981

It is shown that every Killing tensor (KT) identifies a solution for the equation of geodesic deviation (e.g.d.). The general integral of the above equation is also explicitly exhibited in a class of static spherically symmetric space-times found by Kimura.

1. INTRODUCTION

The fundamental role of the equation of geodesic deviation (e.g.d.) in general relativity is connected with the analysis of tidal forces, the focusing effect of gravity (Hawking and Ellis, 1973), the study of Green's functions for perturbation equations, and the evaluation of the first and higher derivatives of the world function and of the parallel propagator (Peters, 1975). In spite of this, there are few attempts to calculate the general integral of the e.g.d., because of the complexity of the differential equations involved.

In this paper we analyze the relations between Killing tensors (KTs) and solutions of the e.g.d. In particular, we develop a systematic method for the evaluation of solutions of the e.g.d., depending on the knowledge of the KTs of the given metric.

[1]Work done under the auspices of the National Group for Mathematical Physics of C.N.R.

0020-7748/82/0500-0391$03.00/0

As an application, the e.g.d. is then solved in closed form in a class of static spherically symmetric space-times found by Kimura (Kimura, 1976; 1977; 1979).

2. EQUATION OF GEODESIC DEVIATION AND KILLING TENSORS

In a given space-time manifold referred to local coordinates x^a ($a = 1,2,3,4$) consider a geodesic of local equations $x^a = x^a(s)$, where s is an affine parameter. The e.g.d. may be written in the standard form (Misner et al., 1973)

$$V^b V^c \nabla_b \nabla_c W_a + R_{ab}{}^c{}_d V^b W_c V^d = 0 \tag{1}$$

where $V^a = dx^a/ds$ is the tangent vector to the geodesic.

Let $W_{(\alpha)a}$ ($\alpha = 1,\ldots,8$) be eight solutions of equation (1). The fields $W_{(\alpha)a}$ are said to be independent if the 8×8 matrix with columns $(W_{(\alpha)1} \cdots W_{(\alpha)1}, (d/ds)(W_{(\alpha)1}), \ldots (d/ds)(W_{(\alpha)1}))$ is nonsingular. It follows from standard theorems on systems of ordinary differential equations (Hartman, 1964) that the general integral of equation (1) is obtained as a linear combination with constant coefficients of eight independent solutions $W_{(\alpha)a}$. We shall now deal with the problem of determining independent solutions of equation (1) by the use of geometric methods.

In general, it may be easily shown that the fields V_a and sV_a are solutions of equation (1). Moreover, it is also known that every Killing vector admitted by the given metric is a solution of the e.g.d. (Manoff, 1979). The above property of Killing vectors may also be extended to KTs in the sense clarified by the following theorem.

Theorem 1. Let $K_{aa_1 \cdots a_p}$ be a KT of order $(p+1)$. Then the field W_a defined as

$$W_a = K_{aa_1 \cdots a_p} V^{a_1} \cdots V^{a_p} \tag{2}$$

is a solution of the e.g.d. (1).

Proof. Let us write the definition of KT in the form

$$\nabla_b K_{aa_1 \cdots a_p} = -\big(\nabla_a K_{a_1 \cdots a_p b} + \cdots + \nabla_{a_p} K_{baa_1 \cdots a_{p-1}}\big) \tag{3}$$

Evaluating the $V^c \nabla_c$ derivative of both sides of equation (3) and transvecting with $V^b V^{a_1} \cdots V^{a_p}$, we obtain, in view of the fact that $V^c \nabla_c V^a = 0$,

$$(p+1) V^b V^c \nabla_b \nabla_c \left(K_{aa_1 \cdots a_p} V^{a_1} \cdots V^{a_p} \right)$$
$$= -V^b V^c V^{a_1} \cdots V^{a_p} \nabla_c \nabla_a K_{a_1 \cdots a_p b} \tag{4}$$

To complete the proof, it is sufficient to apply the commutation rule for covariant differentiation and the definition of KT in the right-hand side of equation (4). Q.E.D.

In order to obtain a better understanding of the significance of Theorem 1, it is convenient to make the following comments.

In the first place, as far as the applicability of the theorem is concerned, it is to be noticed that whenever $K_{aa_1 \cdots a_p}$ is redundant, i.e., it is a linear combination of symmetrized products of lower-order KTs, the field (2) depends linearly on the solutions of the e.g.d. identified by the lower-order KTs. For example, in the Schwarzschild space-time, the above theorem may possibly be used to find solution of the e.g.d. independent of the Killing vectors of the metric, if and only if $p+1>3$, because the Schwarzschild metric does not admit nonredundant KTs of order two and three (Ikeda and Kimura, 1972; Caviglia and Zordan, 1981).

As a second remark, we want to emphasize that a converse of Theorem 1 holds in the following weakened form.

Theorem 2. Let $K_{aa_1 \cdots a_p}$ be a totally symmetric tensor field of order $p+1$ such that every field W_a defined by equation (2) is a solution of the e.g.d. (1). Then $K_{aa_1 \cdots a_p}$ satisfies the relation

$$\nabla_c \nabla_{(b} K_{aa_1 \cdots a_p)} = 0 \tag{5}$$

Proof. Using the Ricci identities and the symmetries of the field $K_{aa_1 \cdots a_p}$ the e.g.d. may be written in the equivalent form

$$V^c V^{a_1} \cdots V^{a_p} V^b \nabla_b \left[(p+1) \nabla_c K_{aa_1 \cdots a_p} + \nabla_a K_{ca_1 \cdots a_p} \right]$$
$$= V^c V^{a_1} \cdots V^{a_p} V^b \nabla_a \nabla_b K_{ca_1 \cdots a_p} \tag{6}$$

Hence, it follows from equation (6) and the arbitrariness of V^a that

$$\nabla_b \nabla_{(c} K_{aa_1 \cdots a_p)} + \nabla_c \nabla_{(a_1} K_{aa_2 \cdots a_p b)} + \cdots$$
$$+ \nabla_{a_p} \nabla_{(b} K_{aca_1 \cdots a_{p-1})} = \nabla_a \nabla_{(b} K_{ca_1 \cdots a_p)} \tag{7}$$

Subtraction of the term $(p+2)\nabla_a \nabla_{(b} K_{ca_1 \cdots a_p)}$ from both sides of equation (7) yields

$$\left[\nabla_b \nabla_{(c} K_{aa_1 \cdots a_p)} - \nabla_a \nabla_{(b} K_{ca_1 \cdots a_p)}\right] + \cdots$$
$$+ \left[\nabla_{a_p} \nabla_{(b} K_{aca_1 \cdots a_{p-1})} - \nabla_a \nabla_{(b} K_{ca_1 \cdots a_p)}\right]$$
$$= -(p+1)\nabla_a \nabla_{(b} K_{ca_1 \cdots a_p)} \tag{8}$$

Evaluating the difference between equation (7) and the relation obtained from equation (7) by interchange of the indices a and b, it may be easily shown that the expression in the first square brackets of equation (8) vanishes. By repeated application of this procedure, it follows that the left-hand side of equation (8) vanishes identically. Hence, we conclude that equation (5) holds. Q.E.D.

In general, the relation (5) is not equivalent to the definition of a KT $\nabla_{(b} K_{aa_1 \cdots a_p)} = 0$. However, every solution of equation (5) will give a KT of order $p+1$ if the metric of the space-time admits only null parallel totally symmetric tensor fields of order $p+2$. For example, using the Ricci identities, we found that in all vacuum space-times, with the only exception of the Petrov type N, a symmetric tensor of order two identifies a KT if the conditions of Theorem 2 are satisfied.

3. AN APPLICATION TO A CLASS OF SPHERICALLY SYMMETRIC SPACE-TIMES

In this section we shall apply the results of Theorem 1 to the explicit determination of the general integral of the e.g.d. for the nonradial geodesics of a suitable class of spherically symmetric space-time metrics.

Let us consider the line element

$$ds^2 = -(br)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + b^{-1} r^2 dt^2 \tag{9}$$

where b is a positive constant. The family of metrics (9) was found by Kimura (see Kimura, 1976, case IIC) in the discussion of quadratic first integrals of the geodesic equation of motion in static, spherically symmetric space-times (Kimura, 1976; 1977; 1979). The metric (9) admits the Killing vectors

$$X_a dx^a = -r^2 \sin\phi \, d\theta - r^2 \sin\theta \cos\theta \cos\phi \, d\phi \tag{10a}$$

$$Y_a\,dx^a = r^2\cos\phi\,d\theta - r^2\sin\theta\cos\theta\sin\phi\,d\phi \tag{10b}$$

$$Z_a\,dx^a = r^2\sin^2\theta\,d\phi \tag{10c}$$

$$U_a\,dx^a = (r^2/b)\,dt \tag{10d}$$

and the second-order nonredundant Killing tensors K_{ab} and H_{ab} whose nonvanishing components are given by (Kimura, 1976)

$$K_{41} = K_{14} = rt/(2b^2), \qquad K_{22} = -t^2r^4, \qquad K_{33} = -t^2r^4\sin^2\theta$$

$$K_{44} = r^2/b^2 + r^4t^2/b \tag{11}$$

$$H_{41} = H_{14} = r/(2b^2), \qquad H_{22} = -2tr^4, \qquad H_{33} = -2tr^4\sin^2\theta$$

$$H_{44} = 2r^4t/b \tag{12}$$

where the coordinates r, θ, ϕ, t are denoted by x^1, x^2, x^3, x^4, respectively.

It may be shown that the finite equations for a nonradial, timelike, equatorial geodesic are given by

$$r = \alpha\sin(bs+c), \qquad \theta = \pi/2, \qquad \phi = -(b\alpha^2)^{-1}J\operatorname{cotg}(bs+c) + f$$

$$t = -(b\alpha^2)^{-1}E\operatorname{cotg}(bs+c) + d \tag{13}$$

where c, d, E, f, J are constants of integration and $\alpha := (E^2/b - J^2)^{1/2}$. It follows that the tangent field V_a is determined by

$$V_a\,dx^a = -(br)^{-2}\dot{r}\,dr - r^2\dot{\theta}\,d\theta - r^2\sin^2\theta\,\dot{\phi}\,d\phi + b^{-1}r^2\dot{t}\,dt \tag{14}$$

where the superimposed dot denotes the derivative with respect to s. Substituting equations (11), (12), and the components of V^a obtained from equations (13) into equation (2) we obtain two solutions P_a and Q_a of the e.g.d. given by

$$P_a\,dx^a = Et/(2b^2r)\,dr - Jt^2r^2\,d\phi + \left(\tfrac{1}{2}rt\dot{r} + E + Ebt^2r^2\right)/b^2\,dt \tag{15}$$

$$Q_a\,dx^a = E/(2b^2r)\,dr - 2Jtr^2\,d\phi + \left(\tfrac{1}{2}r\dot{r} + 2bEtr^2\right)/b^2\,dt \tag{16}$$

Consider now the fields (10), (15); and (16) evaluated on the geodesic (13) as well as the vectors V_a, given by equation (14), and sV_a. It may be shown by a straightforward but long calculation that these vectors are eight independent solutions of the e.g.d., so that they give rise to the general integral of the e.g.d.

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